ACCESS TO SCIENCE, ENGINEERING AND AGRICULTURE: MATHEMATICS 1

MATH00030

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Contents

7. Integral Calculus

7.1. **Introduction to Integration**.

As was the case with the chapter on differential calculus, for most of this chapter we will concentrate on the mechanics of how to integrate functions. However we will first give an indication as to what we are actually doing when we integrate functions. This can be made rigorous mathematically but in this course we just want to get an intuitive idea of what is going on.

Suppose we want to find the area lying between the graph of a function and the x-axis between two given points a and b. Then one way of doing this would be to approximate this area by the area of rectangles which lie under the graph, as shown in Figure 1. The reason we use rectangles is because it is easy to calculate their area, it is simply their height times their width.

Of course the problem with this approach is that we will usually underestimate the area under the curve since we are not including the area above the rectangles and under the graph. One possible solution would be to make the width of the rectangles smaller and smaller. In this way we would hopefully get a better approximation to the area under the curve. However we can not be sure that this would be the case if we are dealing with a really strange function.

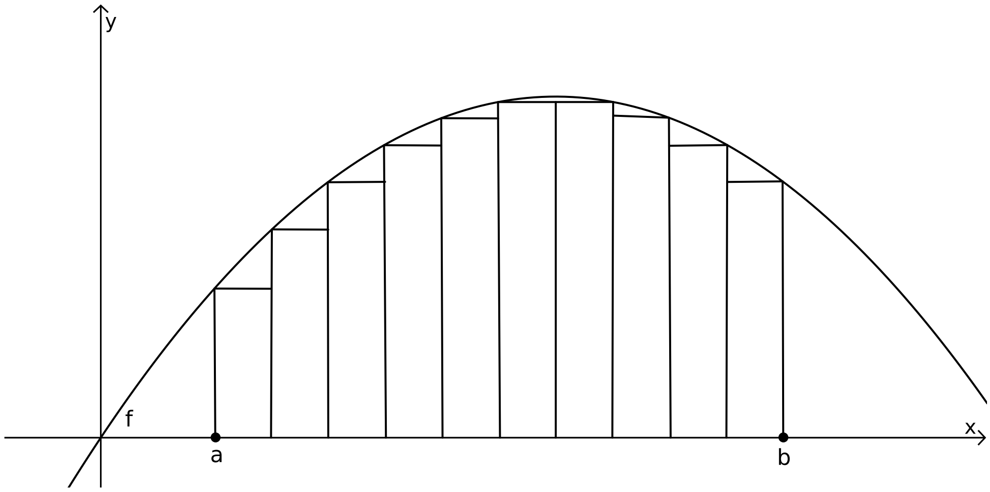

FIGURE 1. An underestimation of the area under the graph of the function f.

Another approach is to overestimate the area by putting the rectangles above the curve as Shown in Figure 2.

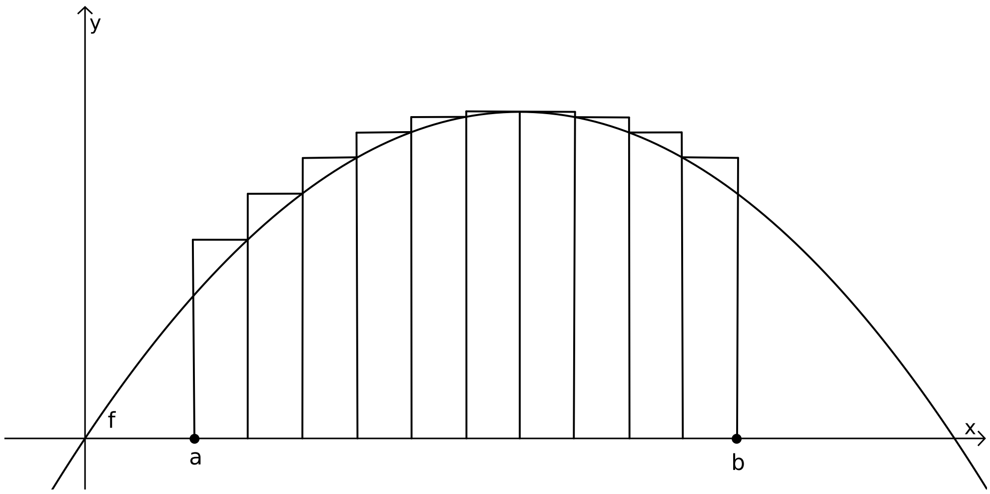

FIGURE 2. An overestimation of the area under the graph of the function f.

You might point out that this doesn't get us any further and you would be correct. Clearly it is no better to have an overestimation of the area. However the clever bit is that we can try and reduce the overestimation by changing the widths of the rectangles and we can try and reduce the underestimation the same way (using different rectangles). If we can get both the overestimation and the underestimation of the area to be 'close' to a given number A then we say that the function f is integrable on the interval $[a,b]$ and we write $\int_a^b f(x)\,dx = A$. In this case the area under the curve *is* A. The number $\int_a^b f(x)\,dx$ has a special name.

Definition 7.1.1 (Definite Integral)**.** If a function f is integrable on the interval $[a,b]$, then the number $\int_a^b f(x)\,dx$ is called the *definite integral* of f from a to b. The function f is called the *integrand*.

In Figures 1 and 2, we have given an example of a function that lies above the x-axis between the points a and b but the area is a 'signed area'. That is if part of the graph of f lies below the x-axis then this area is counted as negative. For example in Figure 3, the integral $\int_a^b f(x)\,dx$ represents the area in red minus the area in green. This means that if we are going to use integrals to calculate areas rather than signed areas, we have to first find which parts of the graph lie above the x-axis and which parts lie below. In the case of Figure 3, the actual area that lies between the graph of f and the x-axis between the points a and b (i.e., the area of the red portion plus the area of the green portion) is $\int_a^c f(x)\,dx - \int_c^b f(x)\,dx$. Note that we have to put a minus sign before the integral $\int_c^b f(x)\,dx$ to allow for the fact that $\int_c^b f(x)\,dx$ gives minus the green area.

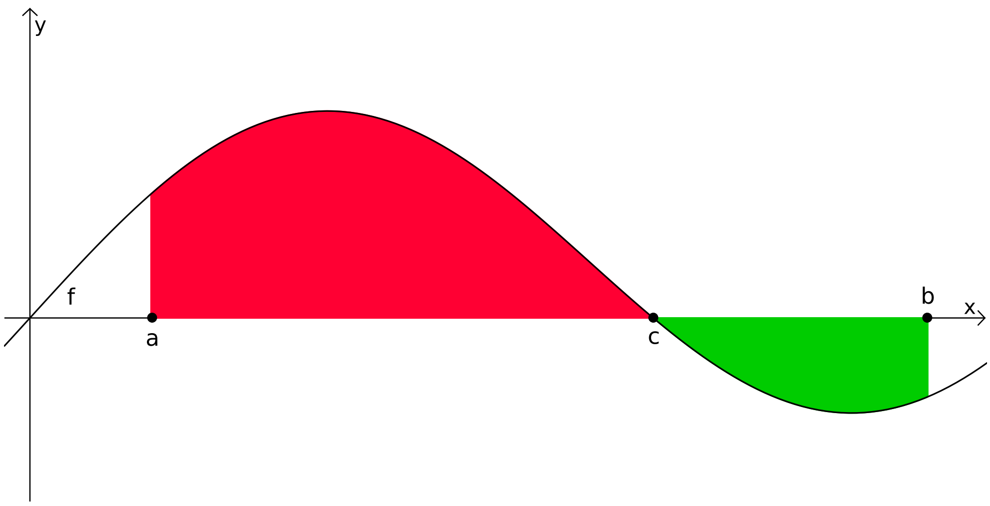

FIGURE 3. Signed area under the graph of the function f.

7.2. The Fundamental Theorem of Calculus.

It is all very well defining an integral as we did in Section 7.1 but in practice it is almost impossible to use this definition to actually calculate areas. Luckily, the Fundamental Theorem of Calculus comes to our rescue. There are several slightly different forms of this theorem that you may meet in your studies but the one we are going to use is the following.

Theorem 7.2.1 (The Fundamental Theorem of Calculus)**.** *Let F and f be functions defined on an interval $[a, b]$ such that f is continuous and such that the derivative of F is f. Then*

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).$$

Remark 7.2.2. Although this result is taught quite early on in your mathematical career, it is a most remarkable and very deep result. It connects two seemingly completely unrelated concepts. Firstly there is the derivative of a function, which gives the slope of a tangent to a curve and then there is the integral of a function, which calculates the area under the curve.

The function F that appears in Theorem 7.2.1 has a special name.

Definition 7.2.3 (Antiderivative)**.** Let F be any function such that the derivative of F is equal to another function f. Then F is said to be an *antiderivative* of f.

Note that the antiderivative of a function is not unique. If F is any antiderivative of f and if c is a constant, then it follows from the sum rule and the fact that the derivative of a constant is zero, that $F + c$ is also an antiderivative of f. However, when using The Fundamental Theorem of Calculus, it doesn't matter if we use F or $F + c$ since $(F + c)(b) - (F + c)(a) = F(b) + c - (F(a) + c) = F(b) - F(a)$. That is the constant will always cancel out.

The function $F + c$, where c is a arbitrary constant, also has a special name.

Definition 7.2.4 (Indefinite integral)**.** Let F be any function such that the derivative of F is equal to another function f and let c be an arbitrary constant. Then $F + c$ is said to be an *indefinite integral* of f and the c is said to be a *constant of integration*. This is written as $\int f(x)\,dx = F(x) + c$. That is, there is no a or b on the integral sign.

Although we have a lot of progress theoretically, we have still not actually calculated any integrals and that is what we will turn our attention to next.

7.3. Some Common Integrals.

As with differentiation, we start with some basic integrals and then use these to integrate a wide range of functions using various rules and techniques. The basic integrals that you will need in this course are collected together in Table 1. The main thing is to learn how to use them rather than learning them off by heart, since this table will be included in the exam paper. Note that in the table, c will stand for an arbitrary constant.

$f(x)$	$\int f(x)\,dx$	Comments
k	$kx + c$	Here k is any real number
x^n	$\frac{1}{n+1}x^{n+1} + c$	Here we must have $n \neq -1$
$\frac{1}{x}$	$\ln(x) + c$	Here we must have $x > 0$
e^{ax}	$\frac{1}{a}e^{ax} + c$	
$\sin(ax)$	$-\frac{1}{a}\cos(ax) + c$	Note the change of sign
$\cos(ax)$	$\frac{1}{a}\sin(ax) + c$	

TABLE 1. Some common integrals

Warning 7.3.1.

(1) As was the case with derivatives, the integrals of $\sin(ax)$ and $\cos(ax)$ are only valid if x is in radians. If x is in degrees then extra constants are needed.
(2) Note that the minus sign occurs with the integral of $\sin(ax)$, rather than the integral of $\cos(ax)$, where it appeared when we were differentiating.

As always, some examples will make things clearer. First of all we will give some indefinite integrals in Table 2.

Remark 7.3.2. If you want to check your answer when you have found an indefinite integral then all you need to do is to differentiate your answer. You should always get back to the function you started with.

In Example 7.3.3 I have given a few examples of definite integrals but really finding the indefinite integral is the hard part. Once you have this, finding the definite integral is just a matter of substituting numbers into the formula. Please do remember however that the value of the antiderivative at the lower limit has to be subtracted from the value of the antiderivative at the upper limit. Also note that when calculating definite integrals, we ignore the constant of integration c since it always cancels out.

$f(x)$	$\int f(x)\,dx$	Comments
0	c	
2	$2x+c$	
-4	$-4x+c$	
$-\pi$	$-\pi x+c$	$-\pi$ is just a number
e	$ex+c$	e is just a number
$\cos(1)$	$\cos(1)x+c$	$\cos(1)$ is just a number
x	$\frac{1}{2}x^2+c$	Since $x=x^1$, $n=1$
x^3	$\frac{1}{4}x^4+c$	Here we take $n=3$
x^{-4}	$-\frac{1}{3}x^{-3}+c=-\frac{1}{3x^3}+c$	Here we take $n=-4$
x^π	$\frac{1}{\pi+1}x^{\pi+1}+c$	π is just a number
x^{-e}	$\frac{1}{-e+1}x^{-e+1}+c$	e is just a number
e^x	e^x+c	Here we take $a=1$
e^{5x}	$\frac{1}{5}e^{5x}+c$	Here we take $a=5$
e^{-7x}	$-\frac{1}{7}e^{-7x}+c$	Here we take $a=-7$
e^{ex}	$\frac{1}{e}\cdot e^{ex}+c=e^{ex-1}+c$	Here we take $a=e$
$\sin(x)$	$-\cos(x)+c$	Here we take $a=1$
$\sin(3x)$	$-\frac{1}{3}\cos(3x)+c$	Here we take $a=3$
$\sin(-2x)$	$\frac{1}{2}\cos(-2x)+c$	Here we take $a=-2$
$\sin(-\pi x)$	$\frac{1}{\pi}\cos(-\pi x)+c$	Here we take $a=-\pi$
$\cos(x)$	$\sin(x)+c$	Here we take $a=1$
$\cos(4x)$	$\frac{1}{4}\sin(4x)+c$	Here we take $a=4$
$\cos(-5x)$	$-\frac{1}{5}\sin(-5x)+c$	Here we take $a=-5$
$\cos(\pi x)$	$\frac{1}{\pi}\sin(\pi x)+c$	Here we take $a=\pi$

TABLE 2. Some examples of indefinite integrals

Example 7.3.3.

(1) Calculate the definite integral $\int_1^2 x^2\,dx$.

$$\int_1^2 x^2\,dx=\left[\frac{1}{3}x^3\right]_1^2=\frac{1}{3}2^3-\frac{1}{3}1^3=\frac{7}{3}.$$

(2) Calculate the definite integral $\int_0^\pi \sin(2x)\,dx$.

$$\begin{aligned}\int_0^\pi \sin(2x)\,dx &= \left[-\frac{1}{2}\cos(2x)\right]_0^\pi \\ &= -\frac{1}{2}\cos(2\pi) - \left(-\frac{1}{2}\cos(0)\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0.\end{aligned}$$

Note that in this case the integral is zero since the area above the x-axis cancels out the area below the x-axis.

(3) Calculate the definite integral $\int_{-2}^{-1} e^{-4x}\,dx$.

$$\int_{-2}^{-1} e^{-4x}\,dx = \left[-\frac{1}{4}e^{-4x}\right]_{-2}^{-1} = -\frac{1}{4}e^4 - \left(-\frac{1}{4}e^8\right) = \frac{e^8 - e^4}{4}.$$

As expected this integral is positive since $e^x > 0$ for all values of x (i.e., the graph of $f(x) = e^x$ lies above the x-axis).

7.4. **The Sum and Multiple Rules**.

As was the case with differentiation, although the integrals in Table 1 are very useful, we would not get very far if these were the only functions we could integrate. Luckily there are rules that allow us to integrate more complicated functions. The first two of these are almost identical to the equivalent ones for differentiation.

Theorem 7.4.1 (The Sum Rule for Integration)**.** *Let $f\colon (a,b) \to \mathbb{R}$ and $g\colon (a,b) \to \mathbb{R}$, then the definite integral of $f+g$ on the interval $[a,b]$ is given by*

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx,$$

provided the integrals of f and g exist.

All this says is that if we want to integrate a sum of two functions then all we have to do is integrate them separately and add the integrals.

Remark 7.4.2. As you might expect there is an equivalent rule for indefinite integrals:

$$\int (f+g)(x)\,dx = \int f(x)\,dx + \int g(x)\,dx.$$

Note that when you have a sum like this you only need to include one constant of integration. This is since if you add an arbitrary constant to an arbitrary constant you just get an arbitrary constant.

Here are a couple of examples of the use of the Sum Rule.

Example 7.4.3.

(1) Evaluate the definite integral $\int_{-1}^{1} x^4 + e^{-x}\, dx$.

$$\begin{aligned}
\int_{-1}^{1} x^4 + e^{-x}\, dx &= \int_{-1}^{1} x^4\, dx + \int_{-1}^{1} e^{-x}\, dx \\
&= \left[\frac{1}{5}x^5\right]_{-1}^{1} + \left[-e^{-x}\right]_{-1}^{1} \\
&= \frac{1}{5}1^5 - \frac{1}{5}(-1)^5 + (-e^{-1}) - (-e^{1}) \\
&= \frac{2}{5} + e - e^{-1}.
\end{aligned}$$

(2) Find the indefinite integral $\int \frac{1}{x} + \cos(-3x)\, dx$.

Provided $x > 0$ (so that $\int \frac{1}{x}\, dx = \ln(x) + c$),

$$\begin{aligned}
\int \frac{1}{x} + \cos(-3x)\, dx &= \int \frac{1}{x}\, dx + \int \cos(-3x)\, dx \\
&= \ln(x) - \frac{1}{3}\sin(-3x) + c.
\end{aligned}$$

As was the case with differentiation, the second rule that will enable us to integrate a larger range of functions is the Multiple Rule.

Theorem 7.4.4 (The Multiple Rule for Integration)**.** *Let $f\colon (a, b) \to \mathbb{R}$ and let $k \in \mathbb{R}$ (here I will use k instead of c to avoid confusion with the constant of integration c). Then the definite integral of kf over the interval $[a, b]$ is given by*

$$\int_a^b (kf)(x)\, dx = k\int_a^b f(x)\, dx,$$

provided the integral of f exists.

All this says is that if we want to integrate a constant multiple of a function, then all we have to do is first integrate the function and then multiply by the constant.

Remark 7.4.5. Of course, there is a corresponding Multiple Rule for indefinite integrals:

$$\int (kf)(x)\, dx = k\int f(x)\, dx.$$

Here are a couple of examples of how the Multiple Rule works.

Example 7.4.6.

(1) Evaluate the definite integral $\displaystyle\int_1^2 -\frac{1}{2x}\,dx$.

$$\begin{aligned}\int_1^2 -\frac{1}{2x}\,dx &= -\frac{1}{2}\int_1^2 \frac{1}{x}\,dx\\ &= -\frac{1}{2}\left[\ln(x)\right]_1^2\\ &= -\frac{1}{2}\left(\ln(2)-\ln(1)\right)\\ &= -\frac{\ln(2)}{2}.\end{aligned}$$

Note that since the graph of $f(x) = -\dfrac{1}{2x}$ lies below the x-axis on the interval $[1, 2]$, the integral $\displaystyle\int_1^2 -\frac{1}{2x}\,dx$ must be negative.

(2) Find the indefinite integral $\displaystyle\int 3e^{4x}\,dx$.

$$\int 3e^{4x}\,dx = 3\int e^{4x}\,dx = 3\frac{1}{4}e^{4x} + c = \frac{3e^{4x}}{4} + c.$$

Here we just write c rather than $3c$ since three times an arbitrary constant is still just an arbitrary constant.

As you would expect, both the sum and multiple rules can be used at the same time. Here are a couple of examples of this.

Example 7.4.7.

(1) Evaluate the definite integral $\displaystyle\int_{-\pi}^{\pi} 2\sin(3x) - 4e^x\,dx$.

$$\begin{aligned}\int_{-\pi}^{\pi} 2\sin(3x) - 4e^x\,dx &= \int_{-\pi}^{\pi} 2\sin(3x)\,dx + \int_{-\pi}^{\pi} -4e^x\,dx\\ &= 2\int_{-\pi}^{\pi}\sin(3x) - 4\int_{-\pi}^{\pi} e^x\,dx\\ &= 2\left[-\frac{1}{3}\cos(3x)\right]_{-\pi}^{\pi} - 4\left[e^x\right]_{-\pi}^{\pi}\\ &= 2\left[-\frac{1}{3}\cos(3\pi) - \left(-\frac{1}{3}\cos(-3\pi)\right)\right] - 4\left[e^{\pi} - e^{-\pi}\right]\\ &= 2\left[\frac{1}{3} - \frac{1}{3}\right] - 4\left[e^{\pi} - e^{-\pi}\right]\\ &= 4\left(e^{-\pi} - e^{\pi}\right).\end{aligned}$$

(2) Find the indefinite integral $\int -\frac{1}{6x} + 5x^5 \, dx$.

Provided $x > 0$ (so that $\int \frac{1}{x} \, dx = \ln(x) + c$),

$$\begin{aligned}
\int -\frac{1}{6x} + 5x^5 \, dx &= \int -\frac{1}{6x} \, dx + \int 5x^5 \, dx \\
&= -\frac{1}{6} \int \frac{1}{x} \, dx + 5 \int x^5 \, dx \\
&= -\frac{1}{6} \ln(x) + 5 \left(\frac{1}{6} x^6 \right) + c \\
&= \frac{5x^6 - \ln(x)}{6} + c.
\end{aligned}$$

Again note we only have the one arbitrary constant.